Algorithmic Game Theory
Congestion Games

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Based on slides by Alexandros Voudouris

General definition

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- m **resources**: $E = \{1, \dots, m\}$

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- Each player i has a set of strategies $S_i \subseteq 2^E$, each of which is a subset of recourses that the player can use
- A state $\boldsymbol{s} = (s_1, \dots, s_n)$ is an instance of the game, where each player has chosen a particular strategy $s_i \in S_i$

General definition

- The **load** $n_e(\boldsymbol{s})$ of a resource $e \in E$ in a state $\boldsymbol{s}$ is equal to the number of players using e:

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- The **cost** of player i in state $\boldsymbol{s}$ is equal to the total latency that she experiences from all resources that she uses:

$$\mathrm{cost}_i(\boldsymbol{s}) = \sum_{e \in s_i} f_e(n_e(\boldsymbol{s}))$$

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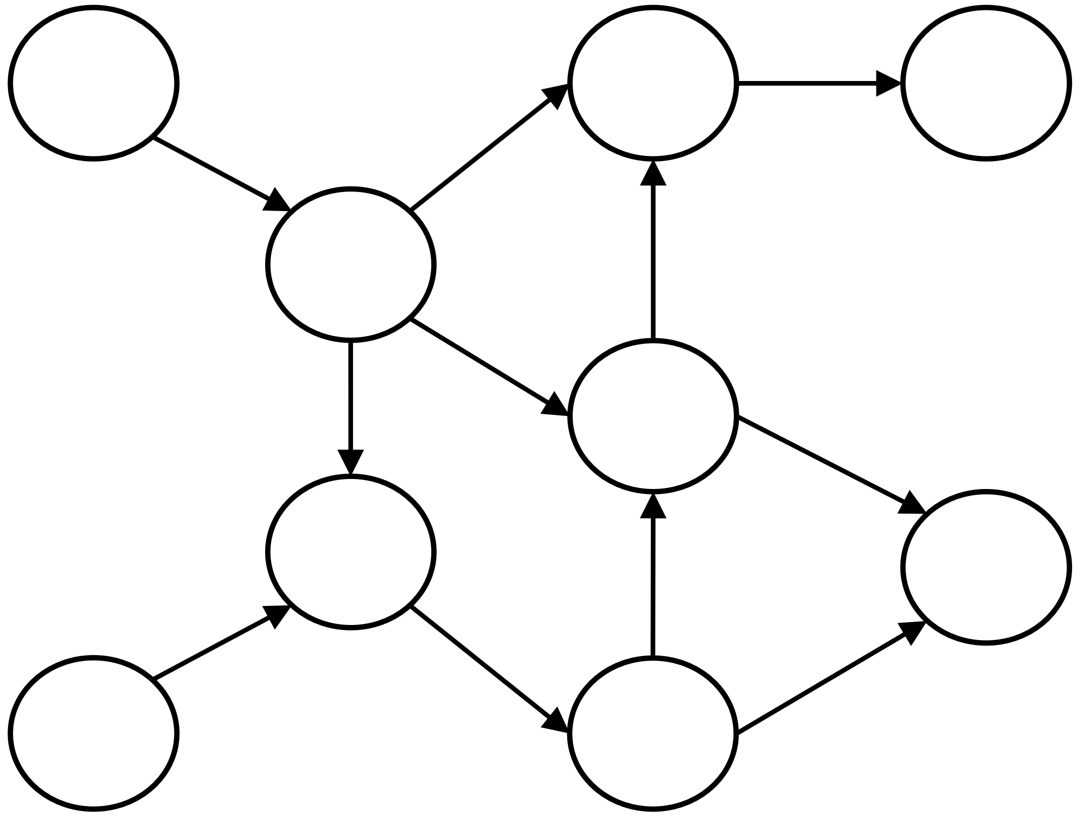
- If all players have the same source node z and the same sink node t, then they all have the same set of possible strategies and the game is symmetric

Network congestion games: example



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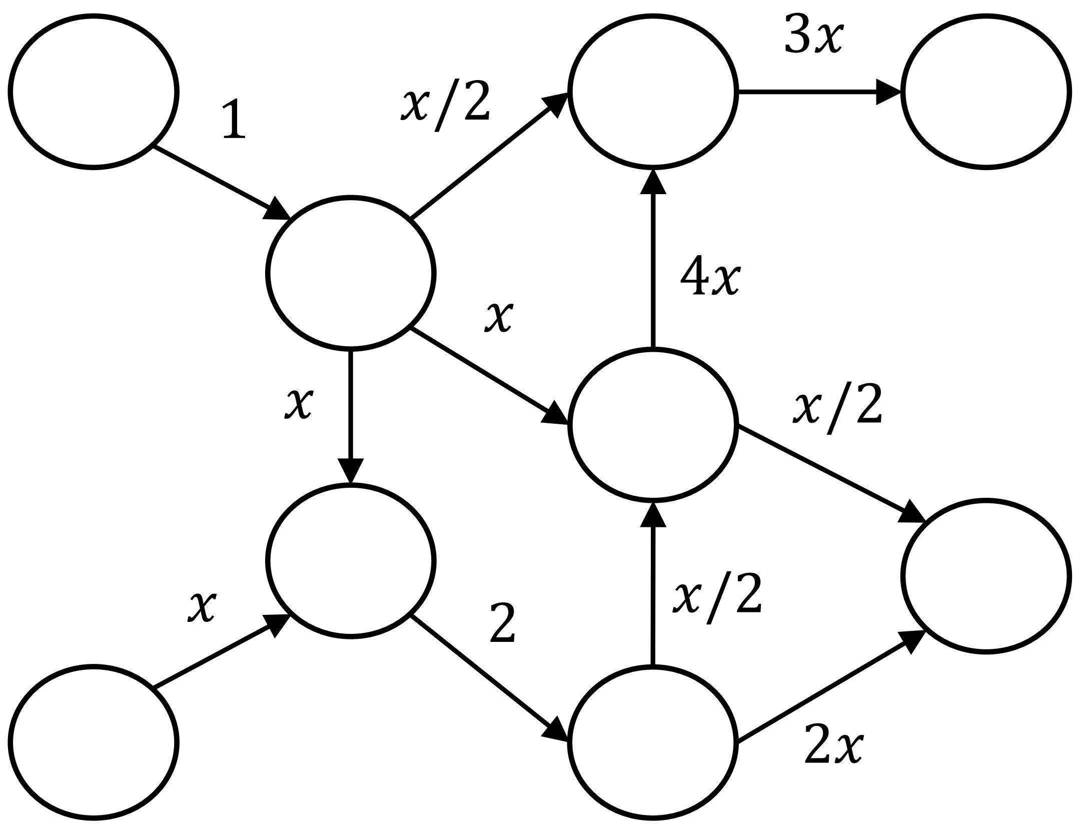

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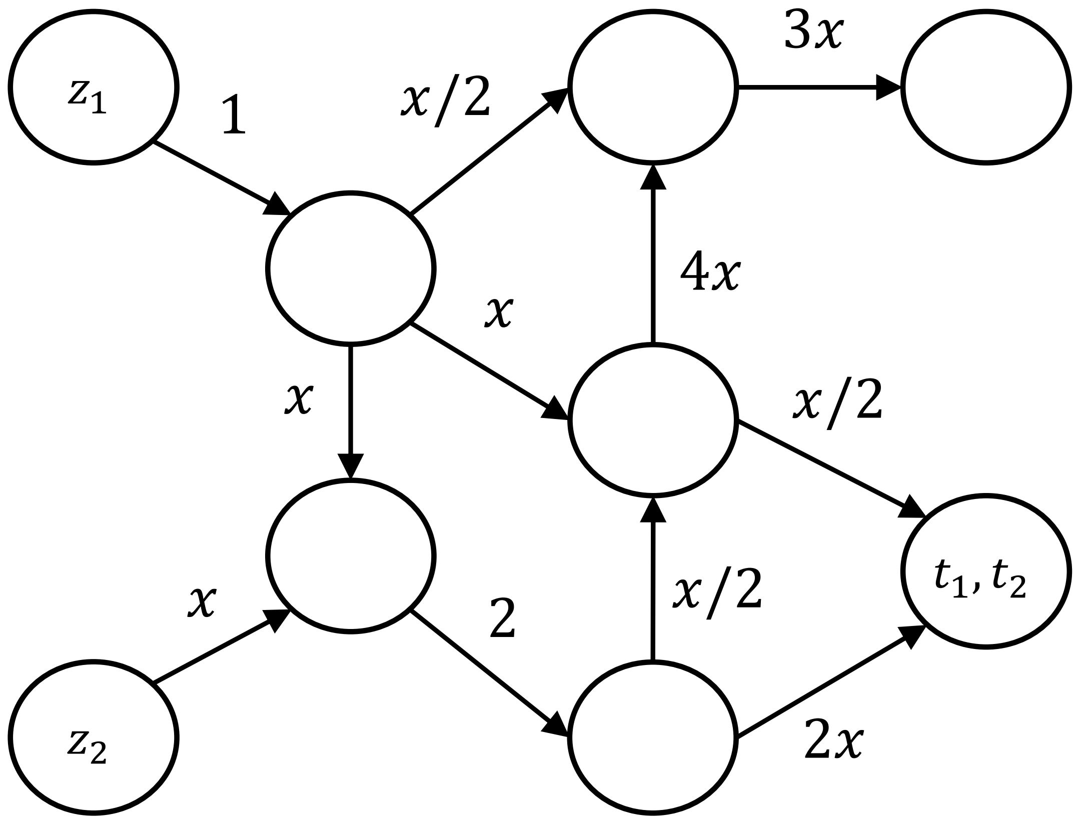

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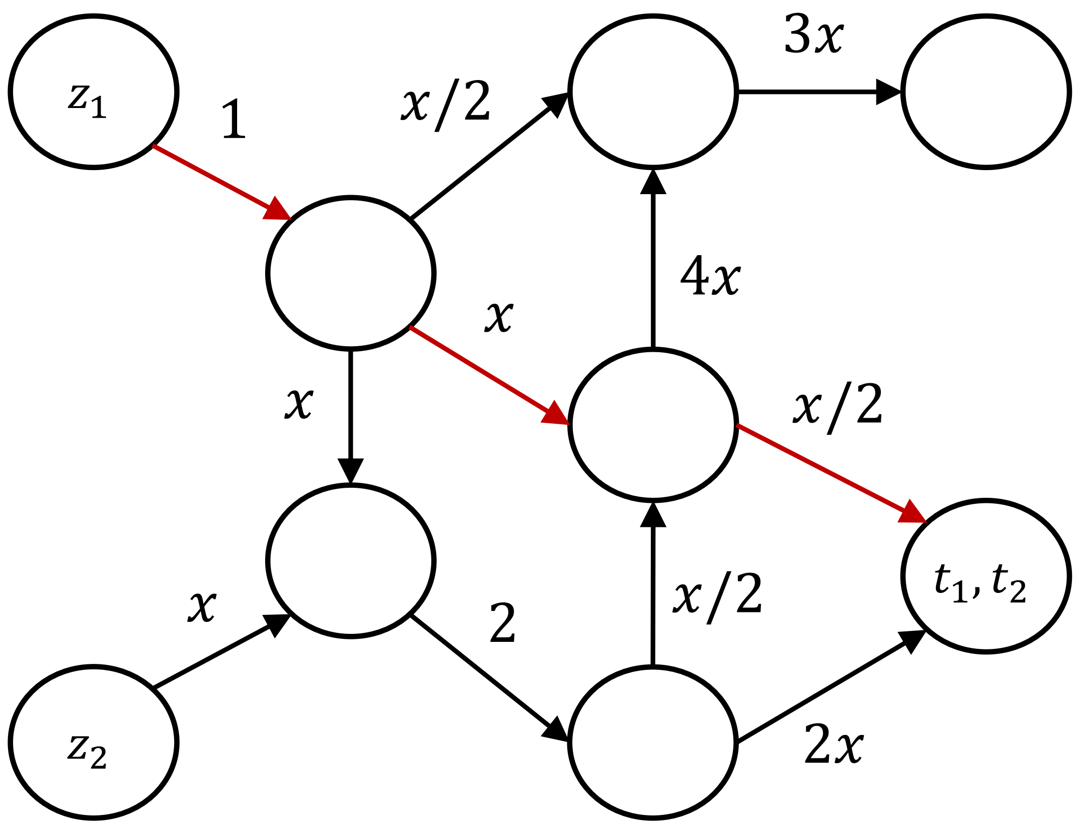

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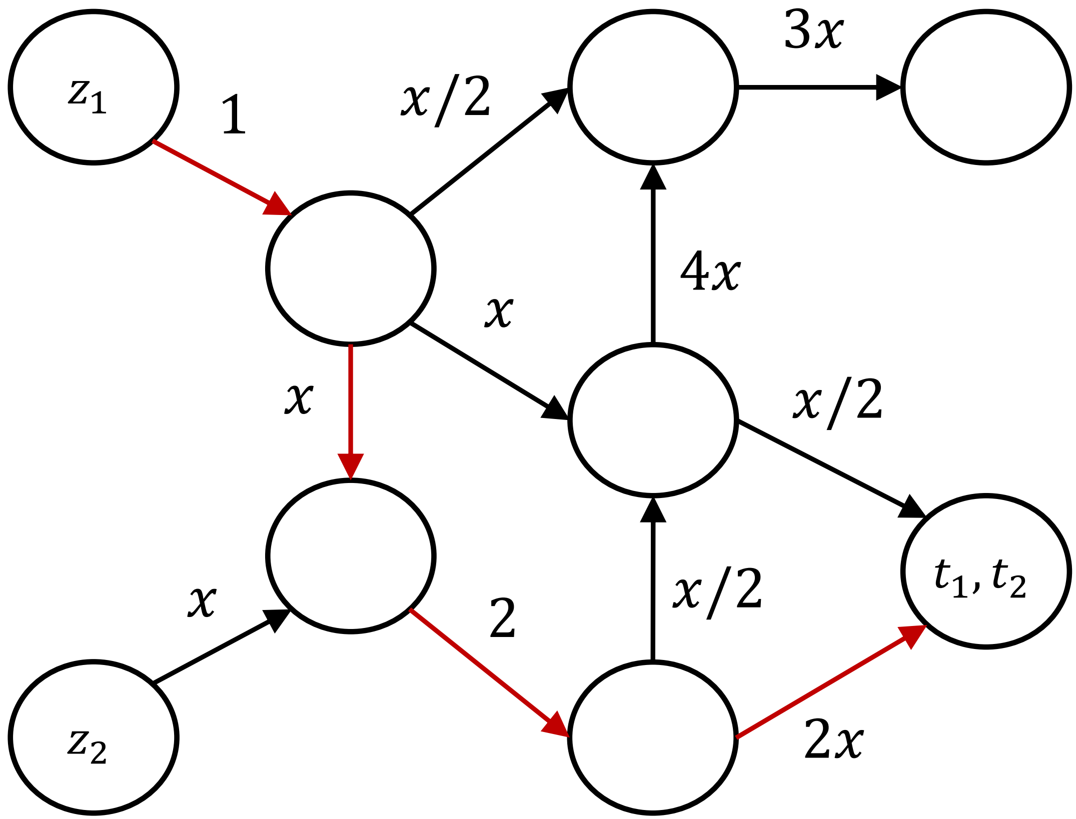

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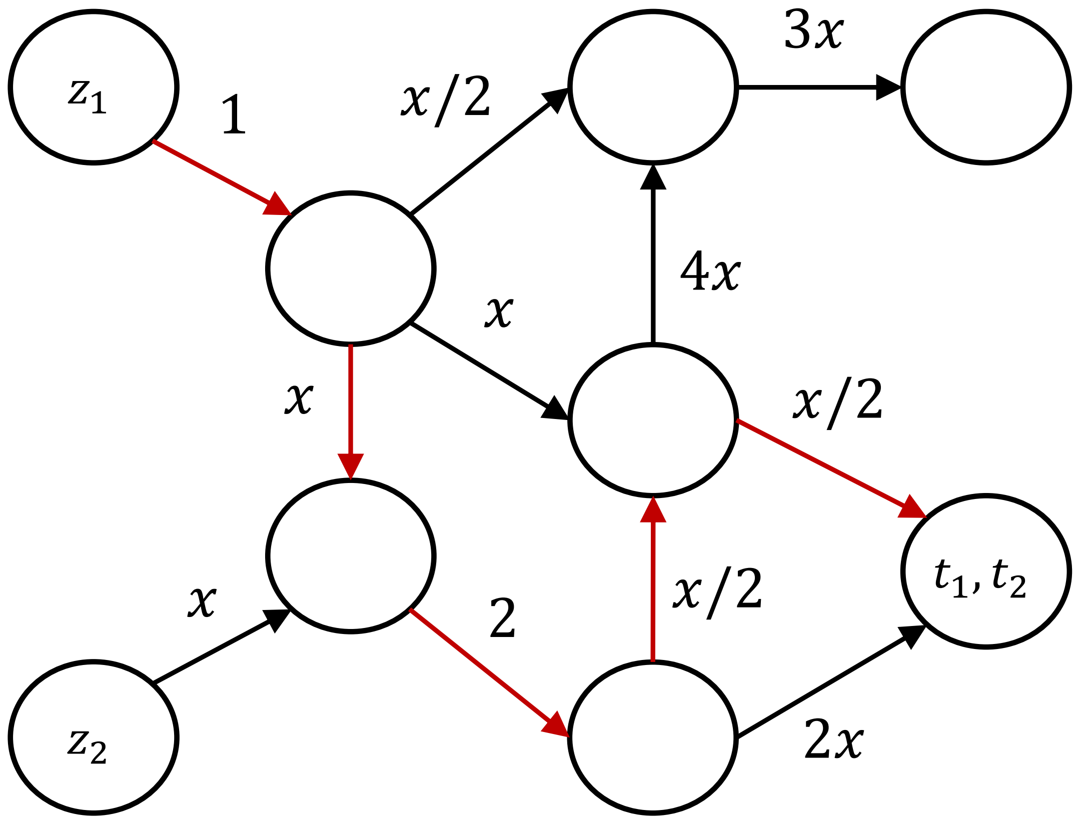

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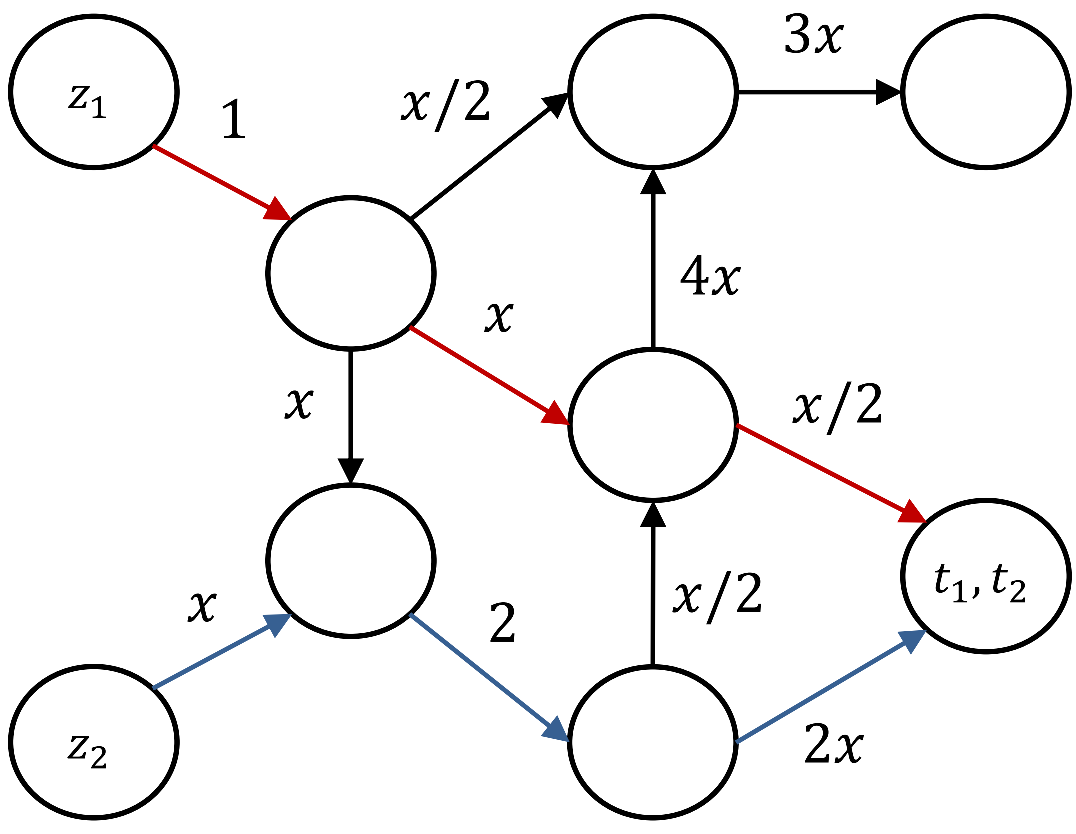

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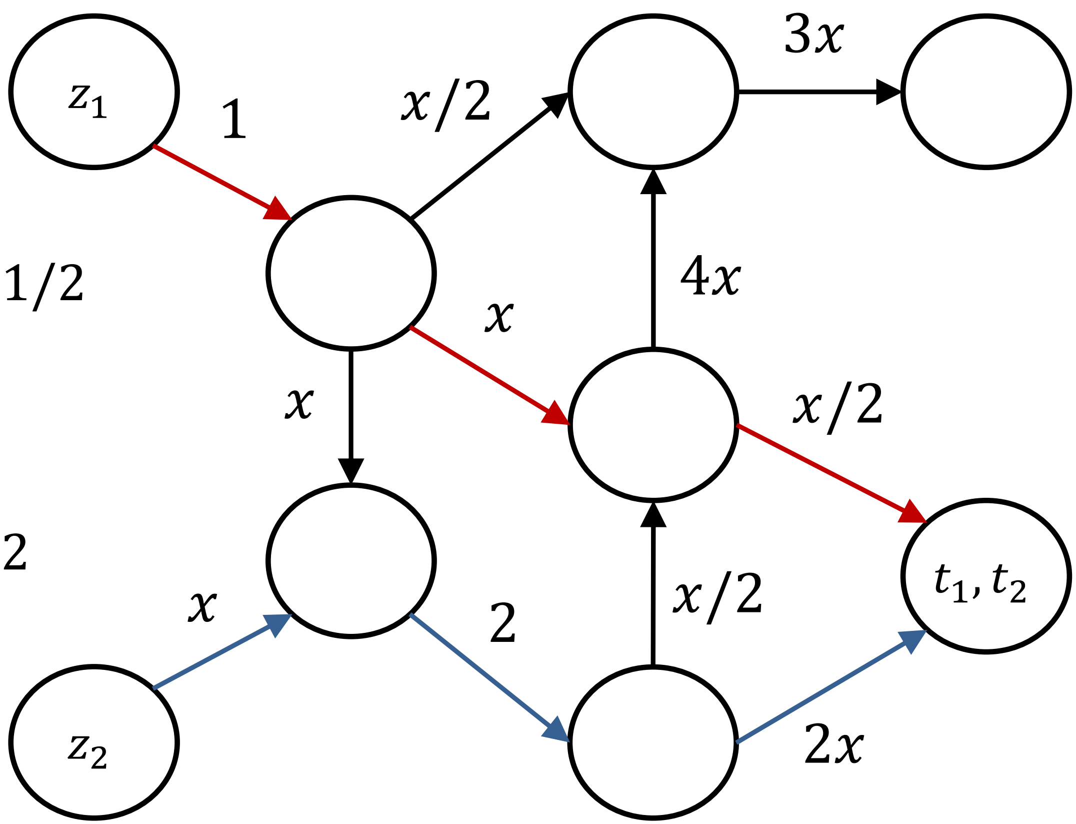

$\text{cost}_1(\boldsymbol{s})$
$= 1 + 1 + 1/2$
$= 2.5$

$\text{cost}_2(\boldsymbol{s})$
$= 1 + 2 + 2$
$= 5$

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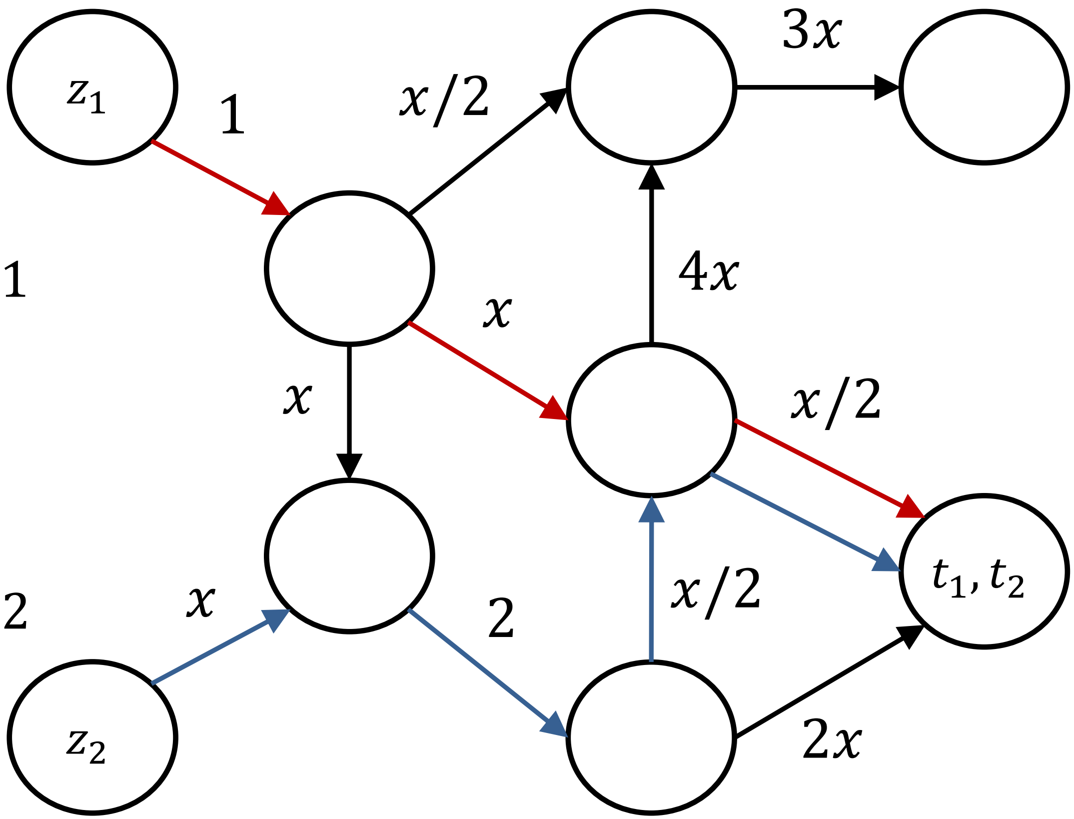

$\mathrm{cost}_1(\boldsymbol{s})$
$= 1 + 1 + 1$
$= 3$

$\mathrm{cost}_2(\boldsymbol{s})$
$= 1 + 2$
$+1/2 + 2/2$
$= 4.5$

Load balancing games

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- The players have jobs that require the same processing time (weight)
- Each player aims to assign her job to a machine so that to minimize the waiting time
- The machines can process in parallel all jobs that have been assigned to them, but have different processing speeds
- If x players choose the same machine of speed v then the cost of each such player is equal to $f_v(x) = x/v$

Load balancing games: example

- Two machines M_1 with speed $v_1 = 1$, and M_2 with speed $v_2 = 2$
- Two players, both with jobs that require 1 hour of processing

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- Two machines M_1 with speed $v_1 = 1$, and M_2 with speed $v_2 = 2$
- Two players, both with jobs that require 1 hour of processing
- If both select M_1 then each of them has a cost of 2
- If both select M_2 then each of them has a cost of 1
- If one selects M_1 and one selects M_2 then the first has cost 1 and the latter has cost $1/2$

	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$
$\boldsymbol{M_1}$	$\mathbf{2, 2}$	$\mathbf{1, 1/2}$
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- Every state besides (M_1, M_1) is an equilibrium

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- What if M_1 has speed $v_1 = 1/2$?

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	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$
$\boldsymbol{M_1}$	$\boldsymbol{4, 4}$	$\boldsymbol{2, 1/2}$
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- It is a dominant strategy for every player to select M_2

Potential functions

- Let Φ be a function which takes as input a state of a game and returns a real value

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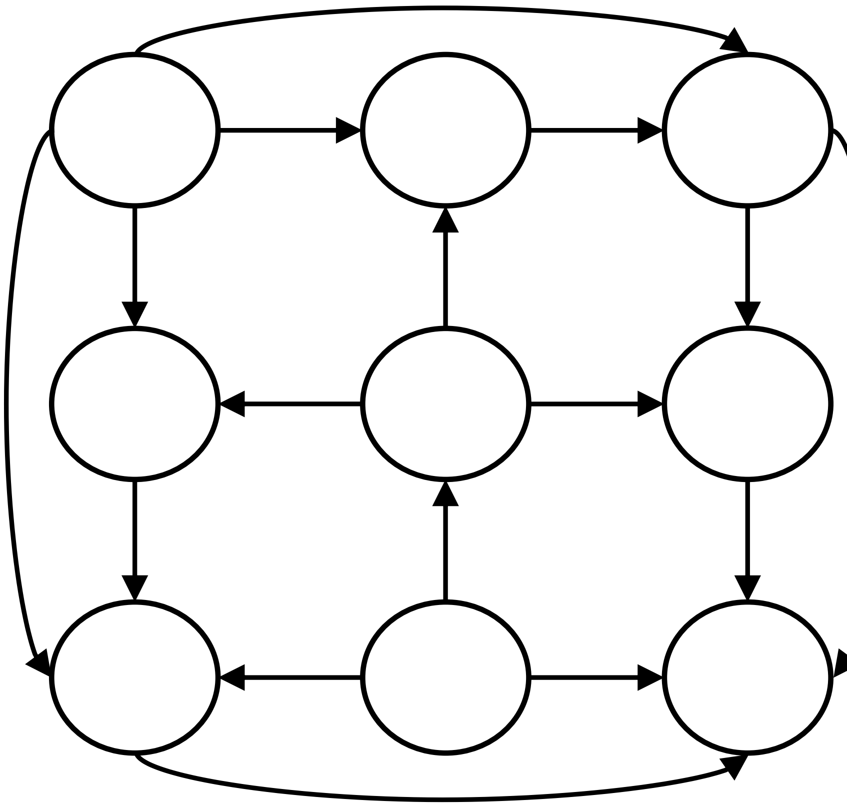
- Let Φ be a function which takes as input a state of a game and returns a real value
- Φ is a **potential function** if for every two states $\boldsymbol{s_1}$ and $\boldsymbol{s_2}$ that *differ on the strategy of a single player* i, the quantities $\Phi(\boldsymbol{s_1}) - \Phi(\boldsymbol{s_2})$ and $\text{cost}_i(\boldsymbol{s_1}) - \text{cost}_i(\boldsymbol{s_2})$ have the same sign:

$$\big(\Phi(\boldsymbol{s_1}) - \Phi(\boldsymbol{s_2})\big)\big(\text{cost}_i(\boldsymbol{s_1}) - \text{cost}_i(\boldsymbol{s_2})\big) > 0$$

Potential functions: example

- Nash dynamics:
  each circle is a state,
  each arrow corresponds to a deviation by a single player who changes strategy to reduce her cost



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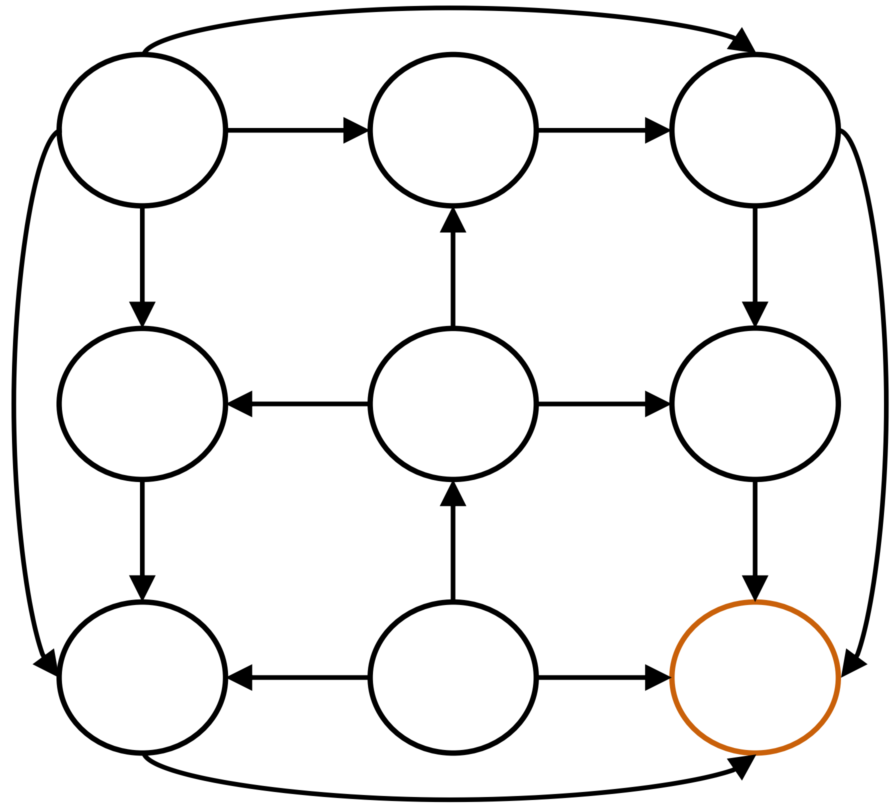

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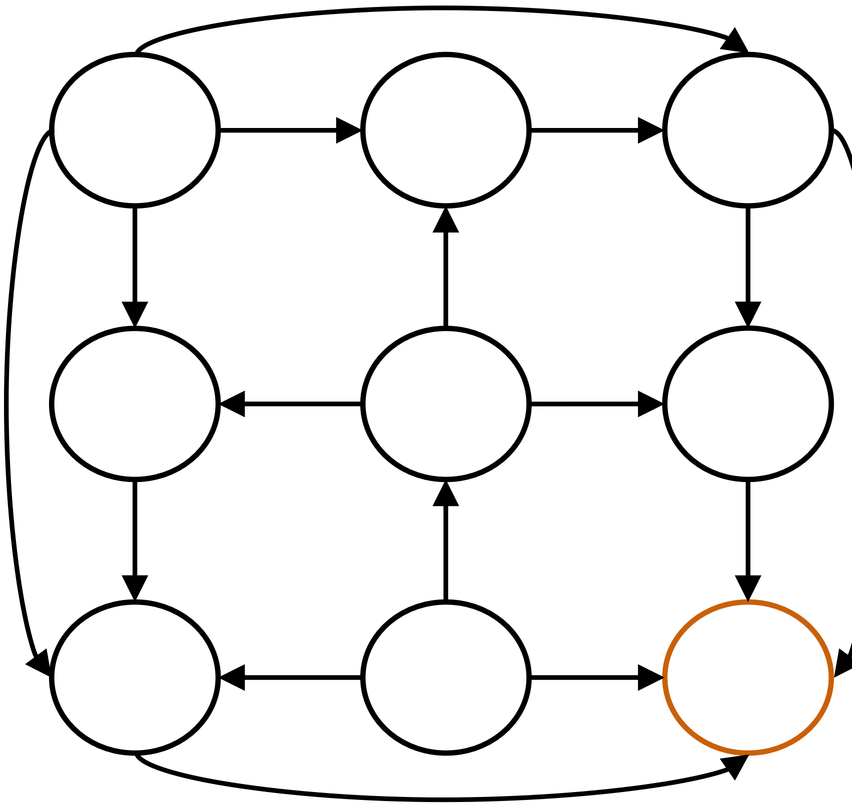

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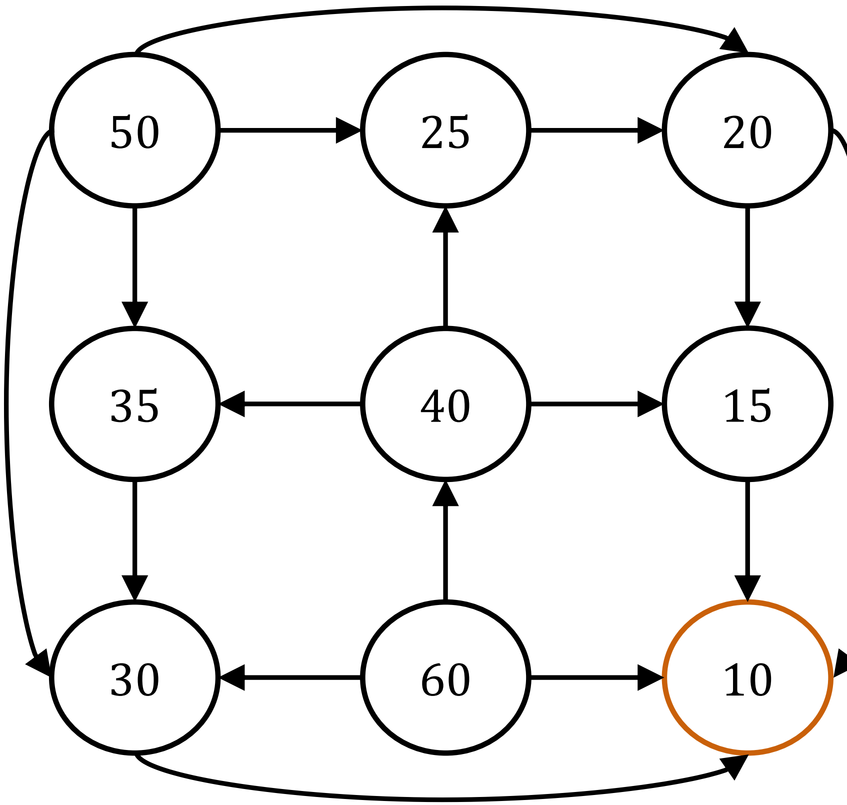

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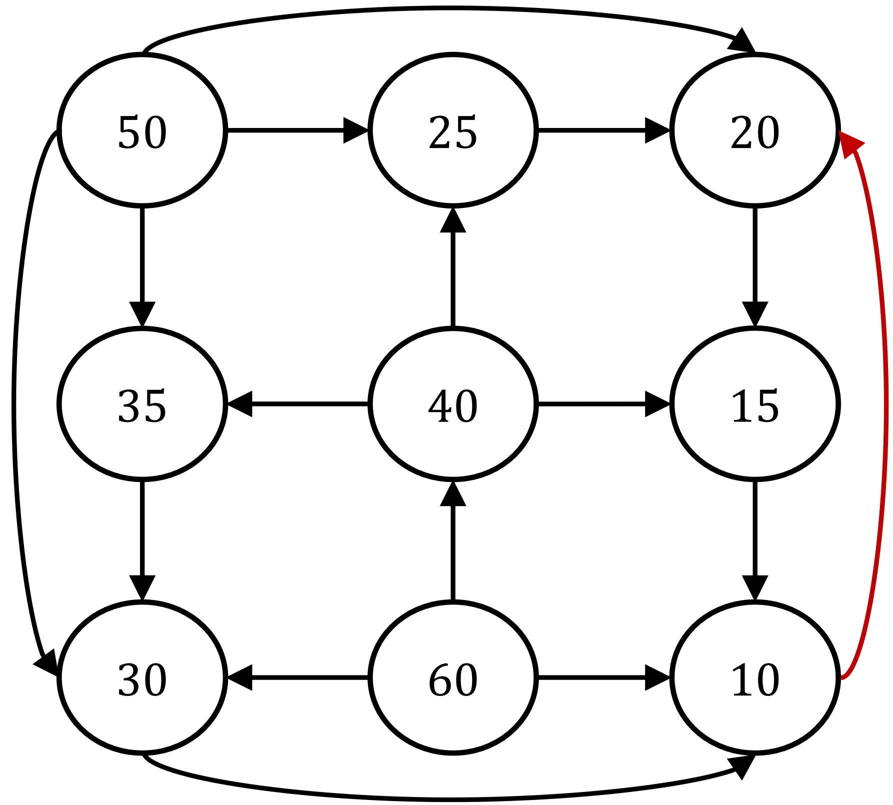

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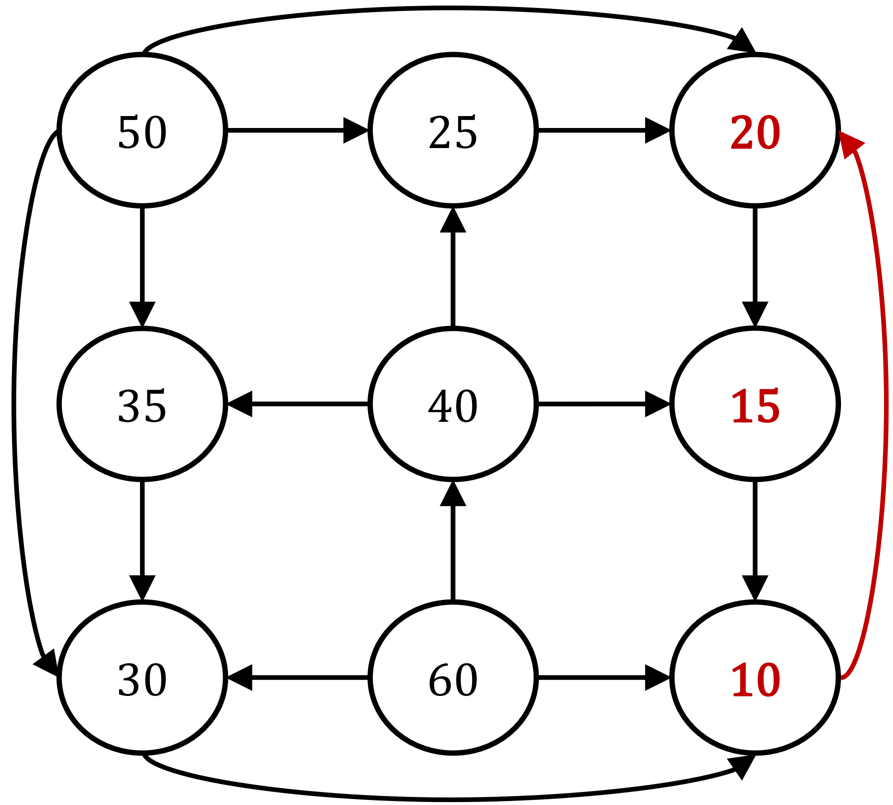
- Let's change the dynamics so that there is no equilibrium
- This is not a valid potential; can we fix this?



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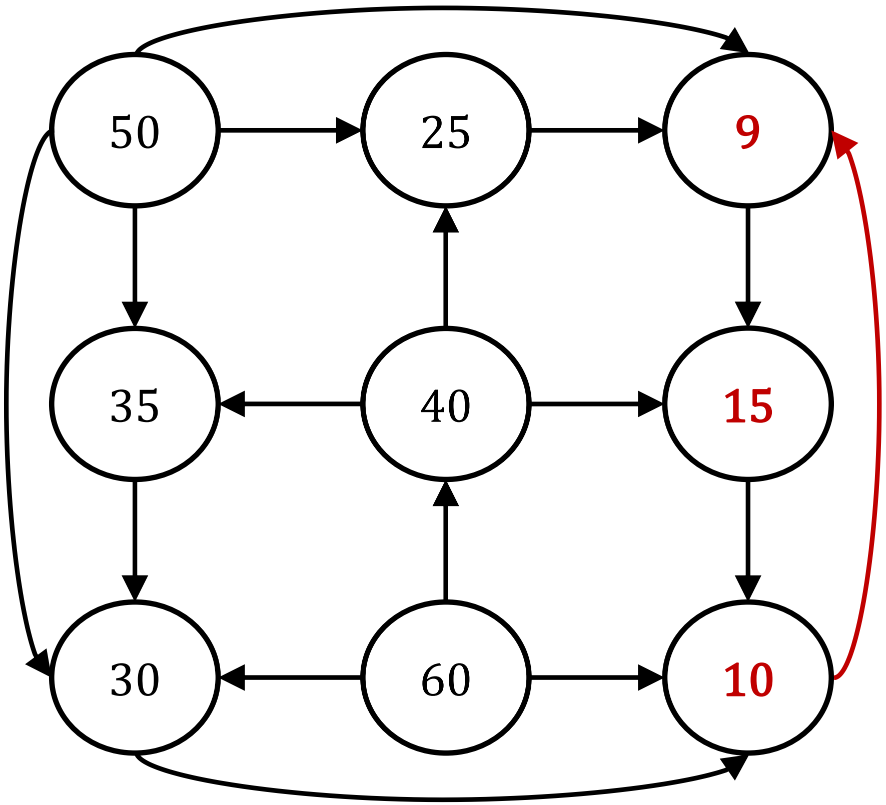

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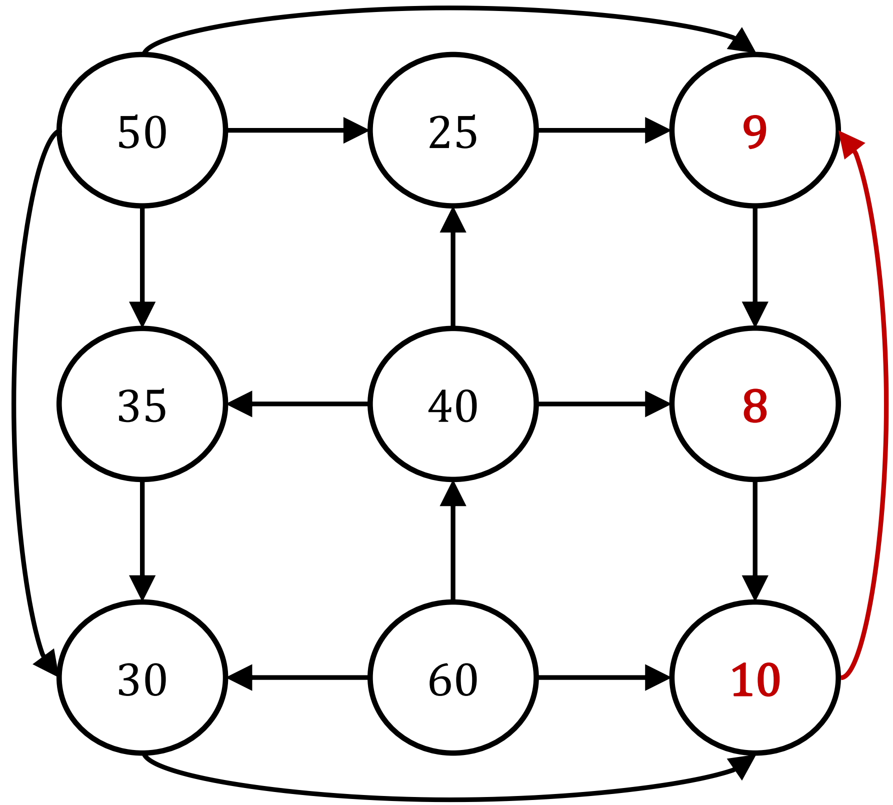

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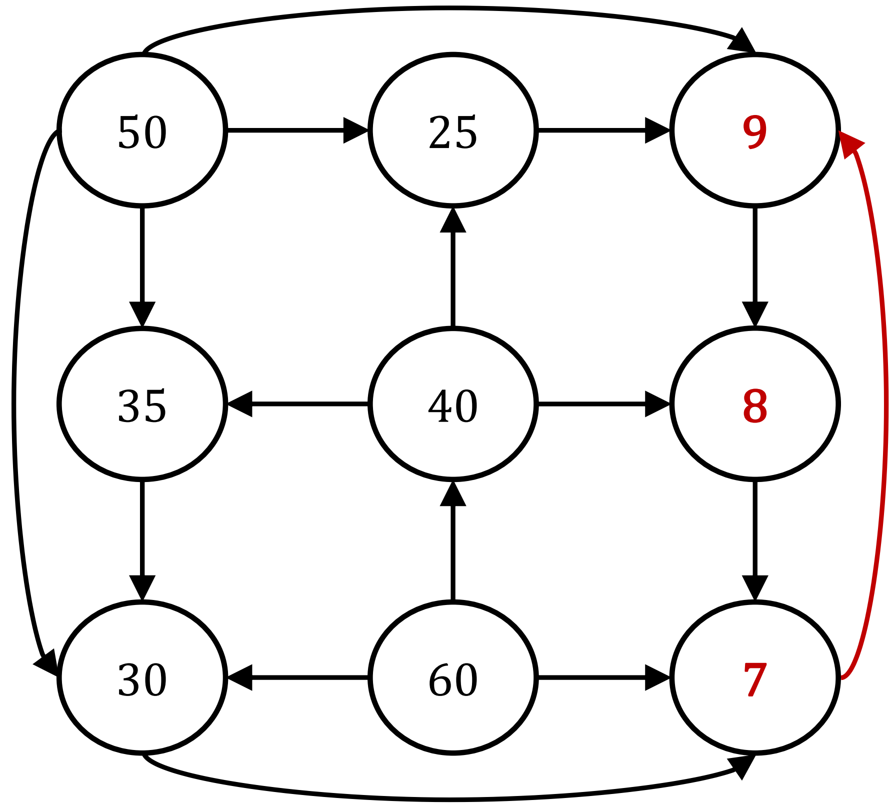

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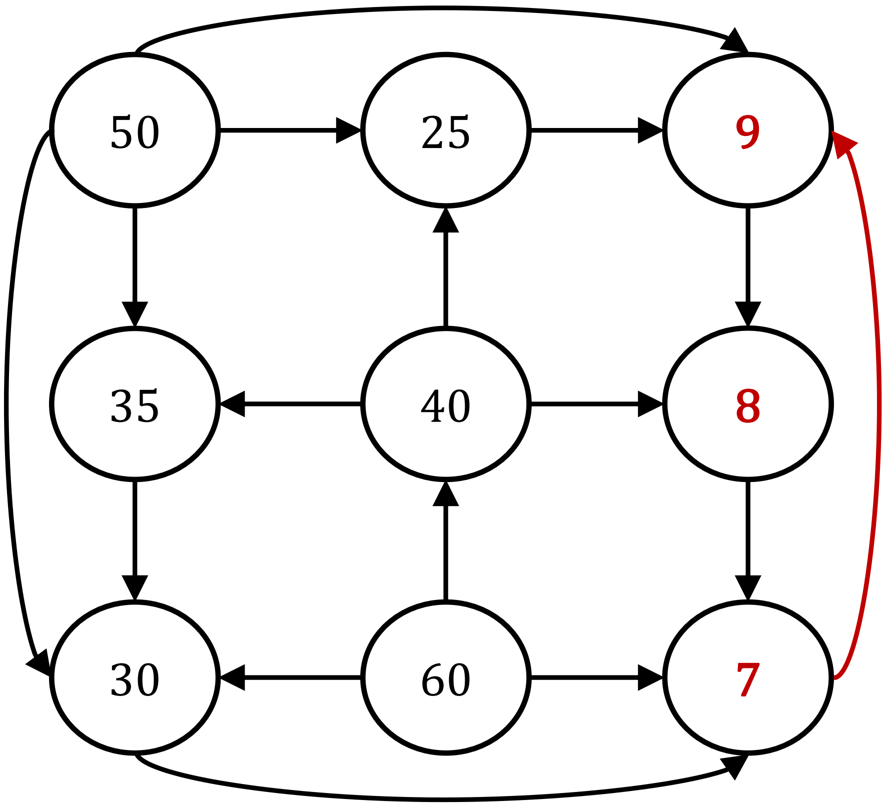

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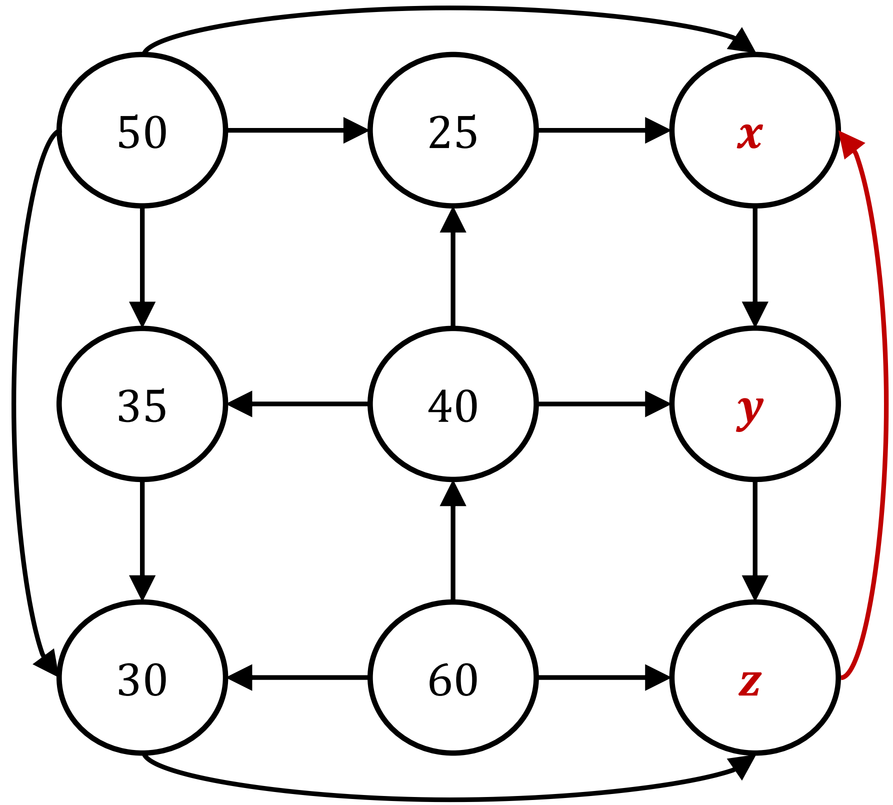
- We must have $x > y > z > x$, a contradiction



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- By the definition of the potential we obtain $\mathrm{cost}_i(\boldsymbol{s}') \geq \mathrm{cost}_i(\boldsymbol{s})$
- Since this holds for every player, $\boldsymbol{s}$ must be an equilibrium $\square$

Rosenthal's function

- For the class of congestion games, Rosenthal [1973] defined the function:

$$\Phi(\boldsymbol{s}) = \sum_{e \in E} \sum_{x=1}^{n_e(\boldsymbol{s})} f_e(x)$$

- Recall:
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- We will show that Rosenthal's function is a potential function for congestion games ⇨ **Every congestion game has at least one pure equilibrium**

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- s_i is the strategy of player i in state $\boldsymbol{s}$
- s_i' is the strategy of player i in state $\boldsymbol{s}'$

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- We partition the set of all resources E into different subsets:
  - $e \notin s_i \cup s_i'$
  - $e \in s_i \cap s_i'$
  - $e \in s_i \setminus s_i'$
  - $e \in s_i' \setminus s_i$

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- $e \notin s_i \cup s_i'$
  - player i does not use e in any of the two states
  - $n_e(\boldsymbol{s}) = n_e(\boldsymbol{s}')$
  - $\sum_{x=1}^{n_e(\boldsymbol{s})} f_e(x) - \sum_{x=1}^{n_e(\boldsymbol{s}')} f_e(x) = 0$

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- $e \in s_i \cap s_i'$
  - player i uses e in both states
  - $n_e(\boldsymbol{s}) = n_e(\boldsymbol{s}')$
  - $\sum_{x=1}^{n_e(\boldsymbol{s})} f_e(x) - \sum_{x=1}^{n_e(\boldsymbol{s}')} f_e(x) = 0 = f_e\big(n_e(\boldsymbol{s})\big) - f_e\big(n_e(\boldsymbol{s}')\big)$

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- $e \in s_i \setminus s_i'$
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  - $n_e(\boldsymbol{s}) = n_e(\boldsymbol{s}') + 1$
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- $e \in s_i' \setminus s_i$
  - player i uses e only in state $\boldsymbol{s}'$
  - $n_e(\boldsymbol{s}) = n_e(\boldsymbol{s}') - 1$
  - $\sum_{x=1}^{n_e(\boldsymbol{s})} f_e(x) - \sum_{x=1}^{n_e(\boldsymbol{s}')} f_e(x) = -f_e\big(n_e(\boldsymbol{s}')\big)$

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- Putting all these together, we have

$$\Phi(\boldsymbol{s}) - \Phi(\boldsymbol{s}') = \sum_{e \in s_i \cap s_i'} \Big(f_e\big(n_e(\boldsymbol{s})\big) - f_e\big(n_e(\boldsymbol{s}')\big) \Big)$$

$$+ \sum_{e \in s_i \setminus s_i'} f_e\big(n_e(\boldsymbol{s})\big) - \sum_{e \in s_i' \setminus s_i} f_e\big(n_e(\boldsymbol{s}')\big)$$

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$$= \sum_{e \in s_i} f_e\big(n_e(\boldsymbol{s})\big) - \sum_{e \in s_i'} f_e\big(n_e(\boldsymbol{s}')\big)$$

Rosenthal's function

- Putting all these together, we have

$$\Phi(\boldsymbol{s}) - \Phi(\boldsymbol{s}') = \sum_{e \in s_i \cap s_i'} \Big(f_e\big(n_e(\boldsymbol{s})\big) - f_e\big(n_e(\boldsymbol{s}')\big) \Big)$$

$$+ \sum_{e \in s_i \setminus s_i'} f_e\big(n_e(\boldsymbol{s})\big) - \sum_{e \in s_i' \setminus s_i} f_e\big(n_e(\boldsymbol{s}')\big)$$

$$= \sum_{e \in s_i} f_e\big(n_e(\boldsymbol{s})\big) - \sum_{e \in s_i'} f_e\big(n_e(\boldsymbol{s}')\big)$$

$$= \mathrm{cost}_i(\boldsymbol{s}) - \mathrm{cost}_i(\boldsymbol{s}') \qquad \square$$

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